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The Two-Nucleon and Three-Nucleon System in Three Dimensions

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We present a brief overview of the three-dimensional formalism that is under development in our group. Using the 3D momentum eigenstates of the nucleon directly, instead of relying on the partial wave decomposition of operators involved in the calculations, allows us to use a very direct approach. This in turn enabled us to successfully tackle a large variety of few-body problems. Our calculation of the two nucleon transition operator and bound state can incorporate a very general form of the two-nucleon potential. Calculations of the three-nucleon bound state can include in addition to the two-nucleon potential also a very general operator form of the three-nucleon force. Recently the 3D formalism is also applied to processes that involve electro-weak probes. Carrying out these calculations for a wide spectrum of two-nucleon and three-nucleon potentials using the classical partial wave approach is unpractical due to the complicated spin structure of the operators. Using the 3D formalism, the calculations can be quickly adapted to test new models.

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1. Formal ingredients

The starting point for our calculations are operator forms of two-nucleon (2N) and three-nucleon (3N) operators and states, developed to take into account symmetry considerations. In these forms the operators and states are typically written as linear combinations of scalar functions and spin, isospin operators. Our first goal is to rewrite the fundamental equations that govern a particular few-body problem into a linear problem acting in a space spanned by the scalar functions.

The 2N potential has the general form [1]:

$$\langle \boldsymbol{p}'|\,\check{V}^{tm_t}\,|\boldsymbol{p}\rangle = \sum_{i=1}^{6} v_i^{tm_t}(|\boldsymbol{p}'|,|\boldsymbol{p}|,\widehat{\boldsymbol{p}}'\cdot\widehat{\boldsymbol{p}})\check{w}_i(\boldsymbol{p}',\boldsymbol{p}). \quad (1)$$

Here $\langle \boldsymbol{p}'|\,\check{V}^{tm_t}\,|\boldsymbol{p}\rangle$ is the matrix element of the 2N force between 2N relative momentum eigenstates $|\boldsymbol{p}\rangle'$, $|\boldsymbol{p}\rangle$ and states $|tm_t\rangle$ where the isospins of the two nucleons are coupled to t with projection m_t. Finally $\check{w}_i(\boldsymbol{p}',\boldsymbol{p})$ are a spin operator and the potential can be reconstructed from a set of scalar functions $v_i^{tm_t}(|\boldsymbol{p}'|,|\boldsymbol{p}|,\widehat{\boldsymbol{p}}'\cdot\widehat{\boldsymbol{p}})$. The form (1) takes into account the partity and time reversal symmetries and assumes no isospin mixing. The same symmetry considerations that led to (1) can be applied to the transition operator satisfying the Lippmann–Schwinger equation

$$\check{t}(E) = \check{V} + \check{G}_0(E)\check{V}\check{t}(E) \quad (2)$$

and $\check{t}(E)$ can be reconstructed from the set of scalar functions $\{t_i^{tm_t}(E;|\boldsymbol{p}'|,|\boldsymbol{p}|,\widehat{\boldsymbol{p}}'\cdot\widehat{\boldsymbol{p}})\}$.

The 2N and 3N bound states have similar decompositions. The general form of the 2N bound state is [2]:

$$\langle \boldsymbol{p}\rangle\,\phi_d m_d = (\mathbb{1}\phi_1(|\boldsymbol{p}|) + (\check{\boldsymbol{\sigma}}(1)\cdot\boldsymbol{p}\check{\boldsymbol{\sigma}}(2)\cdot\boldsymbol{p} - \tfrac{1}{3}\boldsymbol{p}\cdot\boldsymbol{p}\mathbb{1}) \times\phi_2(|\boldsymbol{p}|))\,|1m_d\rangle\,, \quad (3)$$

where $|\boldsymbol{p}\rangle$ is the 3D eigenstate of the relative 2N momentum, $\check{\boldsymbol{\sigma}}(1)$ ($\check{\boldsymbol{\sigma}}(2)$) is a vector of spin operators acting in the space of particle 1 (2), $|1m_d\rangle$ is a spin state with the spins of the particles coupled to the total spin 1 with the projection m_d. Note that the isospin of the system is 0. The main challenge is to calculate the scalar functions $\{\phi_1,\ \phi_2\}$ that can be used to reconstruct the 2N wave function. The general form of the 3N bound state is more complicated [3]. The wave function projected onto a 3N isospin state $\langle(t\tfrac{1}{2})T|$ (where the isospins of two particles are coupled to t and then coupled with the isospin of the third particle to a total isospin T) has the form

$$\langle(t\tfrac{1}{2})T\rangle\,\psi^{3N} = \int \mathrm{d}^3\boldsymbol{p}\,\mathrm{d}^3\boldsymbol{q}\sum_{i=1}^{8}\phi_{tT}^{(i)}(|\boldsymbol{p}|,|\boldsymbol{q}|,\widehat{\boldsymbol{p}}\cdot\widehat{\boldsymbol{q}})\,|\boldsymbol{p}\boldsymbol{q}\rangle \otimes\left([\check{O}_i(\boldsymbol{p},\boldsymbol{q})]^{3\mathrm{Nspin}}[|\chi^m\rangle]^{3\mathrm{Nspin}}\right) \quad (4)$$

and a more detailed description of operators involved in (4) can be found in [3]. Also in this case the 3N wave function can be reconstructed from the set of scalar functions $\{\phi_{tT}^{(i)}\}$ of the magnitudes and the angle between the Jacobi momenta $\boldsymbol{p}$, $\boldsymbol{q}$.

The 3N potential could in principle be treated in a similar fashion as (1). Using the same symmetry considerations could lead to the decomposition of the 3N force into a linear combination of scalar functions and isospin-spin operators. The complicated nature of the 3N potential makes this procedure very difficult. In practical applications we use a more direct representation where the momentum space matrix element of the 3N force for a particular isospin case is represented as a 8×8 dimensional matrix. This greatly increases the size of the numerical work required to carry out the calculations but makes it straightforward to test new force models.

2. Numerical methods

The previous section outlines the basic ingredients of our calculations. The 2N transition operator, 2N bound state and 3N bound state can be reconstructed from sets of scalar functions $\{t_i^{tm_t}\}$, $\{\phi_1,\ \phi_2\}$ and $\{\phi_{tT}^{(i)}\}$,

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respectively. The first step in solving a particular problem involves translating the fundamental equations (Schrödinger, Faddeev, Lippmann–Schwinger) into linear problems on spaces spanned by an appropriate set of functions.

The second step involves solving, sometimes very large, linear (and in the case of bound states eigen-) problem. In some cases the matrix representation of operators involved in the calculation can be produced directly (2N bound state, 2N transition operator [4]). In general, however, this is a very costly and complicated approach. In most practical applications we resort to using the Krylov subspace methods [5] to arrive at a small set (approximately 50) of basis functions that spans the most numerically relevant subspace. This allows us to use standard linear solvers to get the final result.

The size of the numerical task depends on the problem. The simplest calculation involves working out the 2N wave function and can be carried out on a standard PC. Reconstructing the transition operator for a single isospin and initial momentum magnitude case can also be worked out on a standard PC in a reasonable amount of time. Reconstructing the full transition operator for all isospin and initial momentum magnitude cases is, however, a big problem and requires the use of a computing cluster. The most numerically heavy task is the calculation of the 3N wave function. In this case, if the operators involved were to be represented as matrices, the size of these matrices would be of the order 1000000×1000000.

3. Results

Results obtained using the 3D approach were verified and published [4, 6–9]. Here we will present only a few examples of observables for deuteron electro-disintegration (Fig. 1).

As it can be observed in Fig. 2 the 3D results agree very well with classical partial wave results. The partial wave approach, for some cases, requires the incorporation of a large amount of partial waves in order to get a converging result. In such problems the 3D approach is an attractive alternative making it possible to obtain the final result in one move.

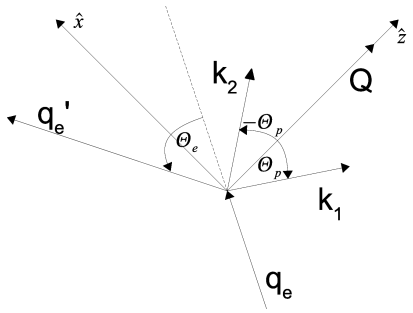

Fig. 1. Kinematical picture for deuteron electro-disintegration. The incoming electron has momentum q_e, the outgoing electron q'_e. Momentum transfer to the 2N system $\boldsymbol{q}_e$, the outgoing electron $\boldsymbol{q}'_e$. Momentum transfer to the 2N system $\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{Q}$ is parallel to the $\widehat{\boldsymbol{z}}$ axis.

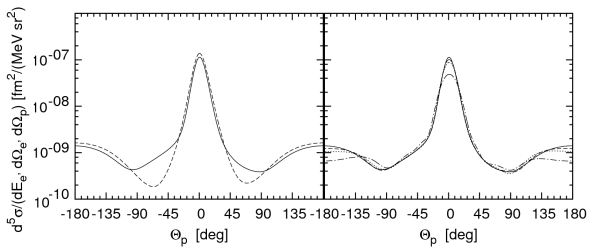

Fig. 2. The unpolarized cross-section as a function of the outgoing proton angle Θ_p from Fig. 1. Negative values of Θ_p correspond to the azimuthal angle $\phi_\mathrm{p} = 0°$ and positive to $\phi_\mathrm{p} = 180°$. On the left panel plane wave results (dashed) are compared with the full calculation (solid). On the right panel the convergence to the full 3D result (solid) is shown for different numbers of partial waves ($j \leq 4$ — dashed-dotted, $j \leq 7$ — dotted, $j \leq 9$ — dashed). Results reprinted from our paper [6] for kinematics *K6*.

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